A METHOD OF ORIENTATING THE PLANE OF A CIRCULAR ORBIT OF A SATELLITE

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1. We consider the motion of an orbiting satellite in a central gravitational force field during the process of changing circular orbits under the influence of an acceleration W, whose vector at any instant is directed along the binormal to the perturbed trajectory. It is known that such a perturbed motion corresponds to a rotation of the osculating orbit [1] relative to the center of gravity in which the shape of the orbit remains unchanged. We are interested in a maneuver resulting in the transfer of the satellite into a given plane of motion. The aim of such a maneuver will be achieved if at the end of the active portion of the flight the plane of the osculating orbit coincides with the given plane.

We will carry out the analysis of the motion of the osculating orbit under the action of the acceleration W.

We fix to the plane P of the unperturbed orbit a right-handed rectangular system of coordinates $Oxyz$ (Fig. 1) with origin at the center of gravity, such that the x-axis is directed towards the point C of the orbit, where the satellite is at the instant t_0 when the maneuver starts, while the y-axis is along a perpendicular to the x-axis in the plane of the orbit and is in the direction of the motion of the satellite.

As a result of the maneuver the triad of axes $Oxyz$ occupies the position $Ox'y'z'$. The new position of the triad relative to the initial position may be described by the Euler angles ψ, θ, φ (Fig. 1). The rate of change of the Euler angles under the action of the acceleration W is in the general case determined by the differential equations of the osculating element [1]; in the particular case of a circular orbit, in which the orbit parameter $p = r$, where r is the magnitude of the radius vector of the satellite, we have

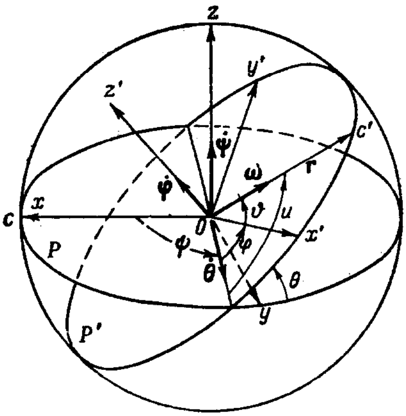

Fig. 1.

$$\frac{d\psi}{dt} = W\sqrt{\frac{r}{\mu}}\frac{\sin u}{\sin\theta}, \quad \frac{d\theta}{dt} = W\sqrt{\frac{r}{\mu}}\cos u$$

$$\frac{d\varphi}{dt} = -W\sqrt{\frac{r}{\mu}}\sin u \cot\theta \qquad (1.1)$$

Here μ is the gravitational constant and u is the argument of the latitude.

We denote by ω the angular velocity vector of the triad $Ox'y'z'$ relative to the axes $Oxyz$. We consider the projection of ω on the axes of a rectangular system of coordinates with origin at the center of mass of the satellite C', consisting of the radius vector of the satellite, its velocity vector, and the binormal to the trajectory. Projecting the angular velocity vector with components $\dot\psi$, $\dot\theta$ and $\dot\varphi$ defined by equations (1.1) onto these axes, we obtain

$$\omega_r = W\sqrt{r/\mu}, \qquad \omega_V = 0, \qquad \omega_n = 0 \qquad (1.2)$$

where ω_r, ω_V and ω_n are the components of ω along the radius vector, velocity vector, and binormal.

The relations (1.2) indicate that in the perturbed motion the triad $x'y'z'$, fixed to the osculating orbit, rotates with angular velocity $\omega = W\sqrt{(r/\mu)}$ about the instantaneous axis which coincides with the radius vector of the satellite.

We define the position of the radius vector r, which is always in the $x'y'$ plane, by the angle ϑ, measured from the x'-axis. For the class of maneuvers considered [2]

$$d\vartheta = \Omega\, dt, \qquad \text{or} \qquad \vartheta = \Omega\,(t - t_0) \qquad \left(\Omega = \frac{1}{r}\sqrt{\frac{\mu}{r}}\right) \qquad (1.3)$$

where Ω is the angular velocity of the satellite in the circular orbit of radius r. From Fig. 1 we have

$$u = \varphi + \vartheta \qquad (1.4)$$

2. We use the methods of the theory of finite rotations for the

determination of the law of variation of the variables ψ, θ and u. Let ω_x, ω_y, ω_z be the projections of the angular velocity vector on the x', y', z' axes, which are fixed to the osculating orbit. Then we have the system [3]

$$\frac{d\alpha}{dt} = \frac{i\omega_z}{2}\alpha + \frac{i}{2}(\omega_x - i\omega_y)\beta, \quad \frac{d\beta}{dt} = -\frac{i\omega_z}{2}\beta + \frac{i}{2}(\omega_x + i\omega_y)\alpha \quad (i = \sqrt{-1}) \tag{2.1}$$

where α, β are the Cayley-Klein parameters, which are expressible in terms of the Rodrigues-Hamilton parameters λ_0, λ_1, λ_2, λ_3 and the Euler angles [3]

$$\begin{aligned} \alpha &= \lambda_0 + i\lambda_3 = \cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2} + i\cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2} \\ \beta &= -\lambda_2 + i\lambda_1 = -\sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2} + i\sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2} \end{aligned} \tag{2.2}$$

For the projections ω_x, ω_y, ω_z we have

$$\omega_x = \omega\cos\vartheta = W\sqrt{r/\mu}\cos\vartheta, \qquad \omega_y = \omega\sin\vartheta = W\sqrt{r/\mu}\sin\vartheta, \qquad \omega_z = 0 \tag{2.3}$$

In the system (2.1) it is convenient to transform to the independent variable ϑ in accordance with the relation (1.3). Carrying out the change of variables, we obtain with the aid of (2.3)

$$\frac{d\alpha}{d\vartheta} = \frac{i}{2}ne^{-i\vartheta}\beta, \qquad \frac{\alpha\beta}{d\vartheta} = \frac{i}{2}ne^{i\vartheta}\alpha \qquad \left(n = \frac{Wr^2}{\mu}\right) \tag{2.4}$$

where n is the lateral transfer thrust at the altitude of the satellite, which we will assume to be constant during the maneuver. We determine the initial conditions of the system (2.4) by assuming that the initial values of the Euler angles at $t = t_0$ are zero, thus

$$\alpha = 1, \qquad \beta = 0 \quad \text{for } \vartheta = 0 \tag{2.5}$$

The system (2.4) reduces to a single linear equation of the second order in α. Its solution for the initial conditions (2.5) takes the form

$$\begin{aligned} \alpha = {} & \frac{1}{2}\left(1 + \frac{1}{\sqrt{n^2+1}}\right)\exp\left[\frac{i}{2}(-1 + \sqrt{n^2+1})\,\vartheta\right] + \\ & + \frac{1}{2}\left(1 - \frac{1}{\sqrt{n^2+1}}\right)\exp\left[-\frac{i}{2}(1 + \sqrt{n^2+1})\,\vartheta\right] \end{aligned} \tag{2.6}$$

Differentiating (2.6) with respect to ϑ and substituting the result into the first equation of the system (2.4), we find the following solution for β

$$\beta=\frac{n}{2\sqrt{n^2+1}}\left\{\exp\left[\frac{i}{2}(1+\sqrt{n^2+1})\,\vartheta\right]-\exp\left[\frac{i}{2}\left(1-\sqrt{n^2+1}\right)\vartheta\right]\right\} \tag{2.7}$$

Separating the real and imaginary parts of (2.6) and (2.7), we have in accordance with (2.2)

$$\begin{aligned}
\lambda_0&=\left(\cos^2\tau+\frac{\sin^2\tau}{n^2+1}\right)^{1/2}\cos\frac{2\rho_0-\vartheta}{2}, &\qquad \lambda_1&=\frac{n}{\sqrt{n^2+1}}\sin\tau\cos\frac{\vartheta}{2}\\
\lambda_3&=\left(\cos^2\tau+\frac{\sin^2\tau}{n^2+1}\right)^{1/2}\sin\frac{2\rho_0-\vartheta}{2}, &\qquad \lambda_2&=\frac{n}{\sqrt{n^2+1}}\sin\tau\sin\frac{\vartheta}{2}
\end{aligned} \tag{2.8}$$

$$\tau=\frac{1}{2}\sqrt{n^2+1}\,\vartheta=\frac{\Omega}{2}\sqrt{n^2+1}\,(t-t_0),\qquad \rho_0=\tan^{-1}\frac{\tan\tau}{\sqrt{n^2+1}}$$

From (2.2) we have

$$\begin{aligned}
\lambda_0&=\cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}, &\qquad \lambda_1&=\sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2}\\
\lambda_3&=\cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}, &\qquad \lambda_2&=\sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2}
\end{aligned} \tag{2.9}$$

Comparing expression (2.8) with expression (2.9), we conclude that

$$\psi+\varphi=2\rho_0-\vartheta,\qquad \psi-\varphi=\vartheta,\qquad \frac{\theta}{2}=\sin^{-1}\frac{n\sin\tau}{\sqrt{n^2+1}} \tag{2.10}$$

It should be noted that a similar formula for θ was obtained in [4]. From the first two equations of the systems (2.10) and (1.4) we find $\psi=\rho_0$, $\varphi=\rho_0-\vartheta$, $u=\rho_0$.

Thus we have the following group of formulas

$$\psi=u=\tan^{-1}\frac{\tan\tau}{\sqrt{n^2+1}},\qquad \theta=2\sin^{-1}\frac{n\sin\tau}{\sqrt{n^2+1}},\qquad \tau=\frac{\Omega\sqrt{n^2+1}}{2}(t-t_0) \tag{2.11}$$

Formulas (2.11) determine the angular coordinates of the satellite in the maneuver for both positive and negative transfer thrusts. However, it is necessary to keep in mind the fact that for $n<0$ the angles ψ, θ and u are measured relative to the line of the descending node. An analysis of the formulas shows that in the perturbed motion the inclination of the osculating orbit to the initial plane does not exceed a value equal to

$$\theta_1=2\sin^{-1}\frac{n}{\sqrt{n^2+1}}=2\tan^{-1}n \tag{2.12}$$

From (2.12), in particular, it follows that the rotation of the plane of the orbit by the angle $\theta=\pi$ is possible only for an infinitely large value of the transfer thrust (an impulsively applied thrust).

The dependence of the angle θ_1 on the transfer thrust is shown in Fig. 2.

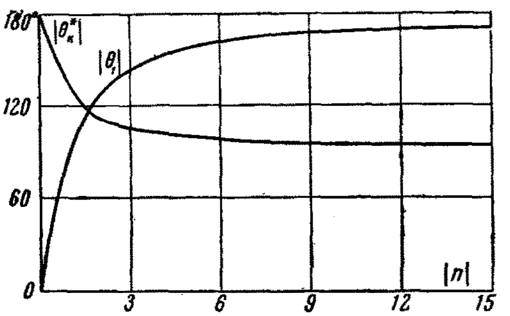

Fig. 2.

The maximum inclination θ_1 is obtained for $\tau = 1/2\ \pi$. The values of all the parameters at this instant are

$$\theta = \theta_1 = 2 \tan^{-1} n, \quad \psi = u = \frac{\pi}{2}$$

$$\vartheta = \frac{\pi}{\sqrt{n^2+1}} \quad \text{for} \quad \tau = \frac{\pi}{2}$$

In [4,5] it is shown that for the maneuver considered here the perturbed trajectory of the satellite is in the plane of a small circle, inclined to the plane of the unperturbed orbit by the angle $\theta_1/2 = \tan^{-1} n$. For $\tau = 1/2\ \pi$ the osculating orbit touches the perturbed trajectory at its highest point with respect to the plane of the initial orbit.

3. In order to obtain angles of rotation greater than $\theta_1 = 2 \tan^{-1} n$ it is necessary, as follows from the second equation of the system (1.1), to keep the sign of $W \cos u$ constant by reversing the direction of the acceleration W at the instant when $\cos u$ passes through zero. The advisability of periodically reversing the direction of the thrust was pointed out in a number of papers; we mention here references [4,6]. We will use the method stated above for investigating the maneuver with a changing sign of the transfer thrust n. We consider the segment of the perturbed trajectory on which the satellite travels with a constant sign of the transfer thrust after the kth change in the sign of n. The sign of the transfer thrust on this segment is defined by the relation

$$\operatorname{sign} n \operatorname{sign} \theta = (-1)^k \qquad \left(k = 1, \ldots, m;\ m = \operatorname{ent} \frac{\pi}{2 \tan^{-1} |n|}\right)$$

Clearly, the system (2.4) is valid for this segment. The initial conditions are the values of the parameters at the instant of the kth change in the sign of n. The values of all parameters corresponding to this instant will be denoted by the index k. On the basis of the results in Section 2, we find

$$u_k = \frac{\pi}{2} + (k-1)\pi, \qquad \tau_k = k\,\frac{\pi}{2}, \qquad \vartheta_k = \frac{k\pi}{\sqrt{n^2+1}} \tag{3.1}$$

$$\psi_k = \frac{\pi}{2}, \qquad \theta_k = 2k \tan^{-1} n(0), \qquad \varphi_k = \frac{\pi}{2} + (k-1)\pi + \vartheta_k \tag{3.2}$$

where $n(0)$ is the transfer thrust at the start of the maneuver.

Substituting (3.2) into the expression (2.2), for $\vartheta = \vartheta_k$ we find

$$\alpha = \cos\frac{\theta_k}{2}\exp\left[\frac{i}{2}(k\pi - \vartheta_k)\right], \qquad \beta = -\sin\frac{\theta_k}{2}\exp\left[-\frac{i}{2}(k\pi - \vartheta_k)\right] \tag{3.3}$$

Solving the system (2.4) for the initial conditions (3.3), we obtain

$$\begin{aligned}
\lambda_0 &= \cos\frac{\theta_k}{2}\left[\cos^2\left(\tau - k\frac{\pi}{2}\right) + \frac{(|n|\tan\Theta^{(k)} - 1)^2}{n^2+1}\sin^2\left(\tau - k\frac{\pi}{2}\right)\right]^{1/2}\cos\left(k\frac{\pi}{2} - \frac{\vartheta}{2} - \rho\right)\\
\lambda_3 &= \cos\frac{\theta_k}{2}\left[\cos^2\left(\tau - k\frac{\pi}{2}\right) + \frac{(|n|\tan\Theta^{(k)} - 1)^2}{n^2+1}\sin^2\left(\tau - k\frac{\pi}{2}\right)\right]^{1/2}\sin\left(k\frac{\pi}{2} + \frac{\vartheta}{2} - \rho\right)\\
\lambda_1 &= \sin\frac{\theta_k}{2}\left[\cos^2\left(\tau - k\frac{\pi}{2}\right) \frac{(|n|\cot\Theta^{(k)} + 1)^2}{n^2+1}\sin^2\left(\tau - k\frac{\pi}{2}\right)\right]^{1/2}\cos\left(\varepsilon - k\frac{\pi}{2} + \frac{\vartheta}{2}\right)\\
\lambda_2 &= \sin\frac{\theta_k}{2}\left[\cos^2\left(\tau - k\frac{\pi}{2}\right) + \frac{(|n|\cot\Theta^{(k)} + 1)^2}{n^2+1}\sin^2\left(\tau - k\frac{\pi}{2}\right)\right]^{1/2}\sin\left(\varepsilon - k\frac{\pi}{2} + \frac{\vartheta}{2}\right)
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
\rho &= \tan^{-1}\left[\frac{|n|\tan\Theta^{(k)} - 1}{\sqrt{n^2+1}}\tan\left(\tau - k\frac{\pi}{2}\right)\right]\\
\varepsilon &= \tan^{-1}\left[\frac{\sqrt{n^2+1}}{|n|\cot\Theta^{(k)} + 1}\cot\left(\tau - k\frac{\pi}{2}\right)\right]
\end{aligned} \qquad \left(\Theta^{(k)} = \frac{|\theta_k|}{2}\right) \tag{3.5}$$

Comparing expressions (3.4) and (2.9), we conclude that

$$\psi + \varphi = k\pi - \vartheta - 2\rho, \qquad \psi - \varphi = 2\varepsilon - k\pi + \vartheta \tag{3.6}$$

$$\sin\frac{\theta}{2} = \sin\frac{\theta_k}{2}\left[\cos^2\left(\tau - k\frac{\pi}{2}\right) + \frac{(|n|\cot\Theta^{(k)} + 1)^2}{n^2+1}\sin^2\left(\tau - k\frac{\pi}{2}\right)\right]^{1/2} \tag{3.7}$$

From the system (3.6) we find

$$\psi = \varepsilon - \rho, \qquad \varphi = k\pi - (\varepsilon + \rho) - \vartheta \qquad \text{or} \qquad u = k\pi - (\varepsilon + \rho)$$

Substituting the expressions (3.5), we obtain finally

$$\psi = \tan^{-1}\left\{\frac{\sqrt{n^2+1}}{|n|(\tan\Theta^{(k)} + \cot\Theta^{(k)})}\left[\frac{1 - n^2 + 2|n|\cot 2\Theta^{(k)}}{n^2+1}\tan\left(\tau - k\frac{\pi}{2}\right) + \right.\right.$$
$$\left.\left. + \cot\left(\tau - k\frac{\pi}{2}\right)\right]\right\} \tag{3.8}$$

$$u = k\pi + \tan^{-1}\left\{\frac{\sqrt{n^2+1}}{2(|n|\cot 2\theta^{(k)}+1)}\left[\frac{1-n^2+2|n|\cot 2\theta^{(k)}}{n^2+1}\tan\left(\tau - k\frac{\pi}{2}\right) - \cot\left(\tau - k\frac{\pi}{2}\right)\right]\right\} \tag{3.9}$$

Formulas (3.7), (3.8) and (3.9) determine the angular coordinates of the satellite during an arbitrary interval of time, in the course of which the transfer thrust is constant and at the start of which the values of the variables satisfy the relations (3.1), (3.2); when the maneuver begins with a negative transfer thrust ($n(0) < 0$) the angles ψ, θ and u determined by the formulas are measured relative to the line of the descending node.

As an illustration, the dependence of the angles ψ, θ and u in the maneuver on τ are shown in Fig. 3 for a transfer thrust $|n| = 0.5$. The initial portion of the maneuver is calculated from formula (2.11) and the latter according to formulas (3.7), (3.8) and (3.9).

Turning to formula (3.7), we reduce it to the form

$$\sin\frac{\theta}{2} = \sin\frac{\theta_k}{2}\left\{1 + \frac{n^2}{n^2+1}\left[\cot^2\frac{|\theta_k|}{2} + \frac{2}{|n|}\cot\frac{|\theta_k|}{2} - 1\right]\sin^2\left(\tau - k\frac{\pi}{2}\right)\right\}^{1/2} \tag{3.10}$$

From (3.10) it follows that the growth of the angle θ with increasing τ proceeds until the trinomial of the second degree in the square brackets remains positive-definite. The positive root of this trinomial, which determines the critical value $\theta_k{}^*$ of the angle θ_k, is equal to

$$\cot\frac{|\theta_k{}^*|}{2} = \frac{-1+\sqrt{n^2+1}}{|n|}, \qquad \text{or} \qquad |\theta_k{}^*| = \pi - \tan^{-1}|n| \tag{3.11}$$

For values $|\theta_k| \geqslant |\theta_k{}^*|$ any further increase in the inclination of the osculating orbit is impossible. As follows from (3.11), the critical case corresponds to motion of the satellite in the plane of a small circle parallel to the initial plane.

For the maneuver illustrated by Fig. 3, the value of θ at the instant of the third change in sign of the transfer thrust exceeds the critical value, hence for $\tau > 270^\circ$ a decrease in the inclination of the osculating orbit is observed.

The dependence of the critical angle $\theta_k{}^*$ on the transfer thrust is shown in Fig. 2.

4. The analysis carried out here enables the synthesis of a maneuver for transferring the satellite from the original plane into a given

plane. For selected values of the lateral transfer thrust, the control of the maneuver reduces to ensuring the required duration of the action of the transfer thrust and the required position of the satellite on the orbit at the instant the maneuver begins.

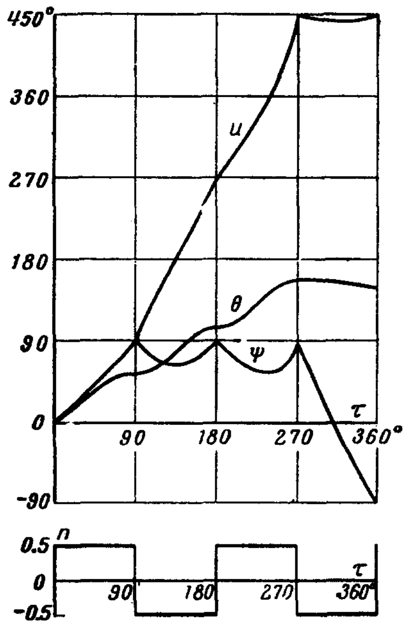

Fig. 3.

The duration of the maneuver is determined with the help of the formula for θ, proceeding from the value of the parameter τ for which the given angle between the planes is attained. In the case of a maneuver in which the direction of the thrust reverses, the required value of τ is determined in two steps. First the number k of sign changes in the transfer thrust is found as the integral part of the ratio

$$\frac{\theta}{2\tan^{-1} n(0)}$$

where θ is the given angle between the planes. Then after θ_k is found, the required value of τ is determined with the aid of formula (3.10), and the duration of the maneuver is

$$t - t_0 = \frac{2\tau}{\Omega\sqrt{n^2+1}}$$

For the known value of τ at the end of the maneuver the shift of the line of nodes of the osculating orbit to the end of the active portion relative to the x-axis of the initial plane is determined. The x-axis coincides with the position of the radius vector of the satellite at the instant when the maneuver begins, hence the value obtained for the angle ψ defines the required position of the satellite at the instant t_0 relative to the given line of intersection of the initial and final orbital planes.

In conclusion we remark that in order to ensure the perpendicularity of the acceleration W to the plane of the osculating orbit, the thrust must rotate with angular velocity $\omega = W\sqrt{(r/\mu)}$ about the radius vector of the satellite.

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